String Picture
of
Bose-Einstein Condensation

S. Bund and Adriaan M. J. Schakel[*]
Institut für Theoretische Physik
Freie Universität Berlin
Arnimallee 14, 14195 Berlin

February 23, 2004

Abstract

A nonrelativistic Bose gas is represented as a grand-canonical ensemble of fluctuating closed spacetime strings of arbitrary shape and length. The loops are characterized by their string tension and the number of times they wind around the imaginary time axis. At the temperature where Bose-Einstein condensation sets in, the string tension, being determined by the chemical potential, vanishes, the system becomes critical, and the strings proliferate. A comparison with Feynman's description in terms of rings of cyclicly permuted bosons shows that the winding number of a loop corresponds to the number of particles contained in a ring.

[*]E-mail: schakel@physik.fu-berlin.de

The first experimental realization of Bose-Einstein condensation (BEC) in a cloud of weakly interacting ^{87}Rb atoms trapped in a magnetic field by a group in Boulder, Colorado [1], and the second definite observation of BEC in a system of Na atoms at MIT [2], turned this field into one of the most active research areas in contemporary condensed matter physics. This recent burst of interest motivated us to consider once more the fundamental properties of BEC.

At present there exist at least three different descriptions of this phenomenon. In the first, and intuitively probably the simplest one, BEC is understood as the condensation of a finite fraction of the total number of particles present in the zero-momentum ground state [3]. The second description, due to Bogoliubov [4], uses the methods of quantum field theory [5]. The formation of a Bose-Einstein condensate is here signaled by a nonzero expectation value of the field describing the particles in the system. The third description, due to Feynman [6], is based on path integrals [7]. BEC is now understood as the occurrence of large rings containing many bosons which are cyclically permuted in imaginary time [8, 9]. Using a mapping onto a certain problem of classical statistical mechanics [10] to numerically evaluate the path integral, Ceperley and Pollock [11] showed that this picture also provides a powerful computational tool with which even a strongly interacting system like superfluid ^{4}He can be accurately described (for a review, see Ref. [12]). The classical system is a chain of beads connected to springs.

In this Letter we discuss an alternative, string picture of BEC. This description is, like Feynman's, based on the spacetime approach to quantum mechanics, i.e., on path integrals. The specific path-integral representation we arrive at was first put forward by Montroll and Ward [13]. Physically, it describes a grand-canonical ensemble of fluctuating closed spacetime strings of arbitrary shape and length. We study the critical behavior of this loop gas close to the temperature T_c where Bose-Einstein condensation sets in, and show that the loops proliferate below T_c. That is to say, the loop gas undergoes a Hagedorn transition at T_c.

The critical behavior we found here is similar to that of vortex loops in superfluids [14] and in type-II superconductors [15]. (For a detailed account of the so-called dual approach to superfluidity and superconductivity in which the vortex loops are the elementary excitations, the reader is referred to the textbook by Kleinert [16].)

Our starting point is the Lagrangian of a nonrelativistic free scalar field theory in Euclidean spacetime $x = (\tau, \mathbf{x})$, with τ the imaginary time [5]:

$$\mathcal{L} = \phi^* \left(\hbar \partial_\tau - \mu - \frac{\hbar^2}{2m} \nabla^2 \right) \phi.$$ (1)

Here, ϕ is the field describing the scalar particles, $\partial_\tau = \partial/\partial\tau$, μ is the chemical potential which accounts for a finite particle number density, and $-\hbar^2 \nabla^2/2m$, with m the mass of the particles, is the kinetic energy operator. Depending on whether the particles are bosons or fermions, the field ϕ is an ordinary commuting or an anticommuting Grassman field.

We adopt the imaginary-time approach to thermal field theory, where the time variable τ is taken to be of finite extent $0 \leq \tau \leq \hbar\beta$, with β the inverse temperature $\beta = 1/k_B T$ (k_B is Bolzmann's constant) [17]. As a result, the energy variable conjugate to τ takes on discrete values. Depending on whether ϕ describes bosons or

fermions, the field is periodic or antiperiodic in τ, $\phi(\hbar\beta, \mathbf{x}) = \pm\phi(0, \mathbf{x})$, and can be expanded in a Fourier series as:

$$\phi(x) = \frac{1}{\hbar\beta} \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{e}^{-ik \cdot x} \phi(k), \tag{2}$$

where $k = (\omega_n, \mathbf{k})$, $k \cdot x = \omega_n \tau - \mathbf{k} \cdot \mathbf{x}$, and ω_n are the discrete Matsubara frequencies,

$$\omega_n = \begin{cases} 2n\pi/\hbar\beta & \text{for bosons} \\ (2n+1)\pi/\hbar\beta & \text{for fermions}, \end{cases} \tag{3}$$

with n an integer.

At finite temperature and density, the thermodynamic properties of the system are specified by the partition function Z. Expressed as a functional integral over the field ϕ it reads [5]:

$$Z = \int \mathrm{D}\phi^* \mathrm{D}\phi \, \exp\left(-\frac{1}{\hbar} \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}^3 x \, \mathcal{L}\right). \tag{4}$$

Since this functional integral is Gaussian, it is easily evaluated to yield

$$\begin{aligned} \ln(Z) &= \ln\left\{ \mathrm{Det}^{-\eta} \left[\left(-i\hbar\omega_n - \mu + \hbar^2 \mathbf{k}^2/2m\right)/\hbar\right] \right\} \\ &= -\eta \mathrm{Tr} \ln\left[\left(-i\hbar\omega_n - \mu + \hbar^2 \mathbf{k}^2/2m\right)/\hbar\right], \end{aligned} \tag{5}$$

where we used the identity Det(A) = exp[Tr ln(A)]. In Eq. (5), $\eta = \pm 1$ for bosons and fermions, respectively, and the trace Tr stands for the integral over Euclidean space-time x as well as the summation over Matsubara frequencies ω_n and the integral over momentum $\mathbf{k}$.

We wish to derive an alternative representation of the system and express Z not as a *functional* integral over the field ϕ, but as a *path* integral over spacetime loops. In the string picture that emerges, the system is characterized by the string tension of the loops and the number of times they wind around the imaginary time axis.

Before considering the partition function, it is expedient to first study the propagator $G(x_1, x_2) = G(x_1 - x_2)$ of the theory,

$$G(x) = \frac{1}{\hbar\beta} \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\hbar}{-i\hbar\omega_n - \mu + \hbar^2 \mathbf{k}^2/2m} \mathrm{e}^{-ik \cdot x}, \tag{6}$$

and write it as a path integral over spacetime trajectories. Employing Schwinger's proper-time method [18], which is based on Euler's form

$$\frac{1}{a^z} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{\mathrm{d}s}{s} s^z \, \mathrm{e}^{-sa}, \tag{7}$$

with $\Gamma(z)$ the Gamma function, we write the right-hand side of Eq. (6) as an integral over proper time s:

$$G(x) = \int_0^\infty \mathrm{d}s \, \mathrm{e}^{\mu s/\hbar} \frac{1}{\hbar\beta} \sum_n \mathrm{e}^{-i\omega_n(\tau-s)} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x} - \hbar\mathbf{k}^2 s/2m}. \tag{8}$$

We next use Poisson's summation formula,

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n \alpha} = \sum_{w=-\infty}^{\infty} \delta(\alpha - w) \tag{9}$$

to replace the summation over the Matsubara frequencies by another one

$$\frac{1}{\hbar\beta} \sum_{n=-\infty}^{\infty} e^{-i\omega_n(\tau-s)} = \sum_{w=-\infty}^{\infty} \eta^w \delta[s - (\tau + w\hbar\beta)]. \tag{10}$$

The physical meaning of the new summation index will become clear shortly. The delta function in Eq. (10) links Schwinger's proper time s to the Euclidean time τ. The propagator now becomes:

$$G(x) = \sum_{w=0}^{\infty} \eta^w \exp\left[\frac{\mu}{\hbar}(\tau + w\hbar\beta)\right] \left[\frac{m}{2\pi\hbar(\tau + w\hbar\beta)}\right]^{3/2} \exp\left[-\frac{m\mathbf{x}^2}{2\hbar(\tau + w\hbar\beta)}\right]. \tag{11}$$

According to the path-integral approach to quantum mechanics [7], this can be written as a sum over all possible spacetime trajectories running from $\mathbf{x}(0) = 0$ to $\mathbf{x}(\tau + w\hbar\beta) = \mathbf{x}$:

$$G(x) = \sum_{w=0}^{\infty} \eta^w G_w(x), \tag{12}$$

with

$$G_w(x) = \int_{\mathbf{x}(0)=0}^{\mathbf{x}(\tau+w\hbar\beta)=\mathbf{x}} D\mathbf{x}(\tau') \exp\left(-\frac{1}{\hbar}\int_0^{\tau+w\hbar\beta} d\tau' L\right). \tag{13}$$

Here, L is the (Euclidean) Lagrangian of a nonrelativistic point particle with mass m in a constant potential $-\mu$:

$$L = \tfrac{1}{2}m\dot{\mathbf{x}}^2(\tau) - \mu, \tag{14}$$

where $\dot{\mathbf{x}}(\tau) = \partial\mathbf{x}/\partial\tau$. Remembering the (anti)periodicity in τ with period $\hbar\beta$, we see that the summation index w, which was introduced in Eq. (10) to replace the summation over the Matsubara frequencies, denotes the number of times a given trajectory winds around the (Euclidean) time axis. The sum over the winding number w in Eq. (12) then implies that the trajectories, which started at $\mathbf{x}(0) = 0$, can wrap arbitrarily many times around the time axis before reaching their end point $\mathbf{x}(\tau + w\hbar\beta) = \mathbf{x}$.

We next turn to the partition function. The two main differences with the propagator are that, first, instead of a factor 1/A in Eq. (6) we have a ln(A) in Eq. (5). And second, the expression for the partition function contains, in contrast to that for the propagator, no x-dependence. Taking these differences into account, we obtain

$$\ln(Z) = \eta V \int_0^{\infty} \frac{ds}{s} e^{\mu s/\hbar} \sum_{n=-\infty}^{\infty} e^{i\omega_n s} \int \frac{d^3k}{(2\pi)^3} e^{-i\hbar\mathbf{k}^2 s/2m}, \tag{15}$$

where we used Euler's form (7) in the limit of small z,

$$\ln(a) = -\int_0^{\infty} \frac{ds}{s} e^{-sa}, \tag{16}$$

3

ignoring an irrelevant additive constant. The integration over spacetime produced a factor $\hbar\beta V$, with V the volume of the system. Following the steps leading to the explicit expression (11) for the propagator, we arrive at the familiar fugacity $\exp(\beta\mu)$ series [3]

$$\ln(Z) = \frac{V}{\lambda^3} \sum_{w=1}^{\infty} \eta^{w+1} \frac{e^{w\beta\mu}}{w^{5/2}}, \tag{17}$$

with

$$\lambda = \hbar\sqrt{2\pi\beta/m} \tag{18}$$

the de Broglie thermal wavelength. In Eq. (17), we omitted the singular zero-temperature contribution corresponding to zero winding number $w = 0$.

The corresponding path-integral representation of $\ln(Z)$ is obtained from Eq. (17) in a similar way as the path-integral representation (12) of the propagator was obtained from Eq. (11). We find:

$$\ln(Z) = V \sum_{w=1}^{\infty} \eta^{w+1} G_w(0)/w, \tag{19}$$

where G_w was expressed as a path integral in Eq. (13). The extra factor $1/w$ stems from the factor $1/s$ in the Schwinger representation (16) of the logarithm, while the zero argument in $G_w(0)$ reflects the absence of any spacetime dependence in the partition function. Explicitly,

$$G_w(0) = \int_{\mathbf{x}(0)=0}^{\mathbf{x}(w\hbar\beta)=0} \mathbf{D}\mathbf{x}(\tau') \exp\left(-\frac{1}{\hbar}\int_0^{w\hbar\beta} d\tau' L\right), \tag{20}$$

implying that this path integral is a sum over all possible *closed* spacetime trajectories, or loops, that return to their starting point after a time $w\hbar\beta$. Since one can start traversing a closed trajectory anywhere along the loop, the extra factor $1/w$ (or, more precisely, $1/w\hbar\beta$) in Eq. (19) can be intuitively understood as arising to prevent overcounting. The arbitrariness of the space coordinate of the starting point yields the volume factor V.

In summary, we expressed the partition function as a path integral over spacetime loops. Each loop in the loop gas can be of arbitrary shape and length, and is characterized by a winding number w, showing how often it wraps around the time axis. The sum over w in Eq. (19) implies that, in principle, this can be arbitrary many times.

With this insight, we return to the explicit expression (17) of the partition function, restricting ourselves to bosons. To be able to carry out numerical studies, we put the system in a box with sides of length D, so that $V = D^3$. As a result, the integration over momentum is replaced by a summation over the discrete variable $\mathbf{k_n} = (2\pi/D)\mathbf{n}$, where $\mathbf{n}$ denotes the integers (n_x, n_y, n_z). This leads in turn to the following change in the propagator (11):

$$\exp\left[-\frac{m\mathbf{x}^2}{2\hbar(\tau + w\hbar\beta)}\right] \to \sum_{\mathbf{n}} \exp\left[-\frac{m}{2\hbar(\tau + w\hbar\beta)}(\mathbf{x} - \mathbf{n}D)^2\right]. \tag{21}$$

4

Instead of Eq. (17), we now obtain as explicit form for the partition function

$$\ln(Z) = \frac{D^3}{\lambda^3} \sideset{}{'}\sum_{w} \frac{e^{w\beta\mu}}{w^{5/2}} W^3 \left(\frac{\pi}{w} \frac{D^2}{\lambda^2} \right), \tag{22}$$

where the prime on the sum will be explained below and

$$W(x) = \sum_{n=-\infty}^{\infty} e^{-xn^2}. \tag{23}$$

The average particle number density $n = \langle N \rangle / V$ in the box is obtained by differentiating $\ln(Z)$ with respect to the chemical potential:

$$n = \frac{1}{\beta} \frac{\partial \ln(Z)}{\partial \mu}. \tag{24}$$

In the following we shall assume this density to be fixed. As numerical value we take $n = 22.22 \text{ nm}^{-3}$, which is the density of liquid ^{4}He at saturated vapor pressure. When varying the size of the system, we always adjust the particle number $\langle N \rangle = N$ in such a way that the density remains constant. For a given volume, Eq. (24) then determines the chemical potential as function of N and β. Figure 1 shows the temperature dependence of μ at fixed particle number $N = 10^7$. We observe that it exhibits critical behavior, tending to zero as

$$\mu(T) \sim -(T - T_{\rm c})^c, \tag{25}$$

with the critical exponent $c = 1.996$. Here, $T_{\rm c}$ is the condensation temperature of a free Bose gas,

$$T_{\rm c} = \frac{2\pi\hbar^2}{k_{\rm B}m} \left[\frac{n}{\zeta(3/2)} \right]^{2/3}, \tag{26}$$

where ζ is the Riemann zeta-function, with $\zeta(3/2) = 2.61238\cdots$. Numerically, taking the ^{4}He value also for the mass parameter, $m = 6.695 \ 10^{-27}$ kg, one finds $T_{\rm c} = 3.145$ K.

The exponent c is related to the correlation length exponent ν and the dynamic exponent z via

$$c = 2\nu z. \tag{27}$$

The dynamic exponent appears here because the strings are embedded in spacetime and parameterized by Euclidean time. (For strings embedded in space, we would have $c = 2\nu$ instead.) The value $c \approx 2$ we found numerically agrees with the expected Gaussian exponents $\nu = 1/2$ and $z = 2$.

Since $\ln(Z)$ is written in Eq. (22) as a sum over the winding number w, also N can be expressed in this way:

$$N = \sideset{}{'}\sum_{w} \langle N_w \rangle, \tag{28}$$

where

$$\langle N_w \rangle = \frac{D^3}{\lambda^3} \frac{e^{w\beta\mu}}{w^{3/2}} W^3 \left(\frac{D^2}{w\lambda^2} \right) \tag{29}$$
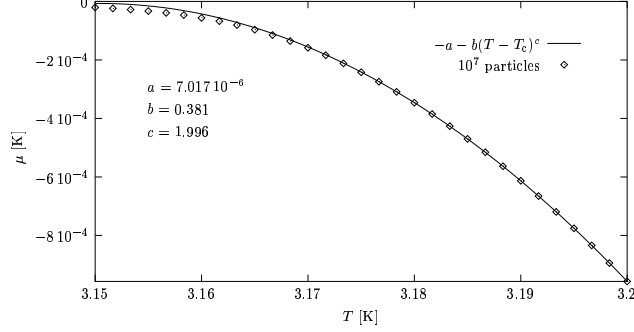
5

Figure 1: Fit of the chemical potential.

denotes the average number of particles contained in loops with winding number w. To obtain a loop interpretation of the total particle number N, note that $\langle N_w \rangle$ is simply related to the average number $\langle L_w \rangle$ of loops with winding number w via

$$\langle L_w \rangle = \frac{\langle N_w \rangle}{w}, \tag{30}$$

so that

$$N = {\sum_w}' w \langle L_w \rangle. \tag{31}$$

This means that the particle number N is given by the total loop "length" measured in units of the characteristic "length" scale $a = \hbar\beta$. We used quotation marks here because the loops are embedded in spacetime and parameterized by Euclidean time, so that a has the dimension of time, not of length.

Since we keep the number of particles in the box fixed, the summations in Eqs. (28) and (31) have the upper bound N; there are no particles available to fill longer loops. To indicate this we decorated the Sigma's with a prime.

The partition function written in terms of loops reads

$$\ln(Z) = {\sum_w}' \langle L_w \rangle. \tag{32}$$

It follows that each loop gives the same contribution to the pressure $P = \ln(Z)/\beta V$, and also to the energy density

$$e = -\left.\frac{\partial(\beta P)}{\partial \beta}\right|_{\beta\mu} = \frac{3}{2}\frac{1}{\beta V}{\sum_w}' \langle L_w \rangle \tag{33}$$

of the system, independent of the loop length or, equivalently, the number of particles contained in the loop.

In the limit of large D, the loop length distribution (30) with Eq. (29) reduces to

$$\langle L_w \rangle = \left(\frac{m\beta}{2\pi}\right)^{3/2}\left(\frac{D}{a}\right)^3\left(\frac{\ell}{a}\right)^{-5/2}\mathrm{e}^{-\beta\sigma\ell}, \tag{34}$$
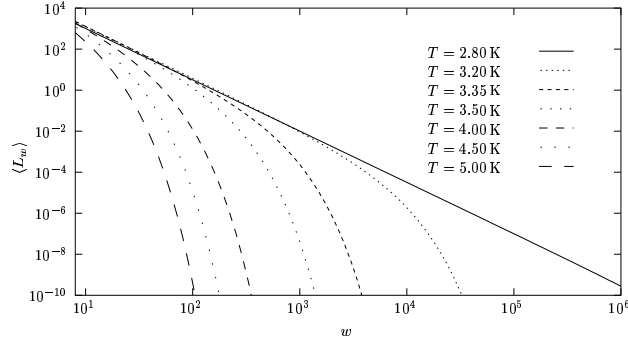
6

Figure 2: Loop length distribution for various temperatures.

where $\ell = wa$ is the "length" of the loop measured in units of $a = \hbar\beta$, and $\sigma = -\mu/\hbar\beta$ is the string "tension". Equation (34) is the typical form for the loop length distribution of Brownian strings [19]. Figure 2 shows this distribution for various temperatures, assuming again liquid ^{4}He values for the mass parameter $m = 6.695\ 10^{-27}$ kg and the (fixed) particle number density $n = 22.22$ nm^{-3}. For $T > T_{\mathrm{c}}$, the chemical potential is negative and the strings have a positive tension. In this temperature regime, we see an exponential decay. When the condensation temperature T_{c} is approached from above, the chemical potential and thus the string tension vanishes. The decay is seen to change from exponential to algebraic. Because their tension vanishes, the strings proliferate. (The temperature where this happens is known in this context as the Hagedorn temperature.) BEC, therefore, corresponds to the appearance of long loops, wrapping arbitrarily many times around the time axis [9].

A similar proliferation of line-like objects we found here, occurs in superfluids and type-II superconductors [16]. The line-like objects in these systems are vortices which are the elementary excitations of the dual approach. In particular, the string tensions and the loop length distributions behave similarly [14, 15]. Only the numerical values of the critical exponents differ as a superfluid and a type-II superconductor belong to the XY universality class, whereas a free Bose gas has Gaussian exponents. Another difference is that a dual transformation interchanges the low- and high-temperature phases, so that the vortices proliferate in the high-temperature phase.

To study numerically the correspondence between BEC and the appearance of long or, for an infinite system, infinite loops, we follow Ref. [14] and distinguish two types of loops: small ones with winding number $w < \sqrt{N}$, and long ones with $w > \sqrt{N}$. The choice $w = \sqrt{N}$ as boundary is motivated by our observation that the average winding number squared $\langle w^2 \rangle$, which gives the compressibility of the system, makes a jump of the order of N at the condensation temperature. Figure 3 shows the temperature-dependence of the average number $\langle N^{\infty} \rangle$ of particles contained in long loops,

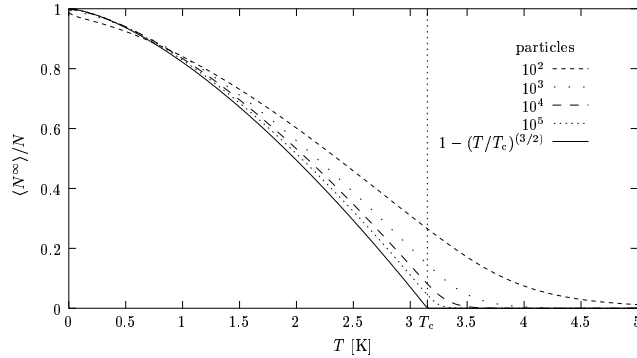$$\langle N^{\infty} \rangle = \sum_{w=\sqrt{N}}^{N} \langle N_w \rangle, \tag{35}$$

7

Figure 3: Fraction of particles contained in long loops.

as fraction of the total number of particles present. Because of Eq. (31), revealing that the particle number in the string picture is nothing but the total loop length measured in units of $a = \hbar\beta$, Fig. 3 can be equivalently interpreted as showing the temperature-dependence of the combined length of long loops as fraction of the total loop length. We see that long loops are exponentially suppressed in the normal phase, implying that here only small loops containing just a few particles are present. The larger the system is, the more suppressed the long loops are. Below the condensation temperature, long loops appear in the system. We see their number increasing with decreasing temperature. At the absolute zero of temperature, virtually all particles are contained in long loops. With increasing particle number N, the curves converge to the solid line given by

$$\frac{\langle N^\infty \rangle}{N}\bigg|_{N\to\infty} = \begin{cases} 1 - (T/T_c)^{3/2} & \text{for } T < T_c \\ 0 & \text{for } T > T_c, \end{cases} \tag{36}$$

implying that also this loop length, like the string tension, exhibits critical behavior at T_c. The right-hand side of Eq. (36) is the well-know expression for the number of condensed particles. From this we conclude that the particles contained in long loops are condensed in the zero-momentum ground state, so that BEC indeed corresponds in the string picture to the appearance of long loops, i.e., to the proliferation of strings.

An interesting question is how many particles are needed for BEC. We investigate this point numerically by calculating the heat capacity for various values for the particle number N (see Fig. 4). We observe that already for a particle number of the order 10, the behavior of the heat capacity at low temperatures closely resembles that of a system with many particles. From this we conclude that about 10 particles suffice for BEC to arise.

In the final part of this Letter, we wish to establish contact with Feynman's path-integral representation of the partition function of a nonrelativistic system and its interpretation in terms of rings of cyclicly permuted bosons [8]. To this end, we recast Eq.
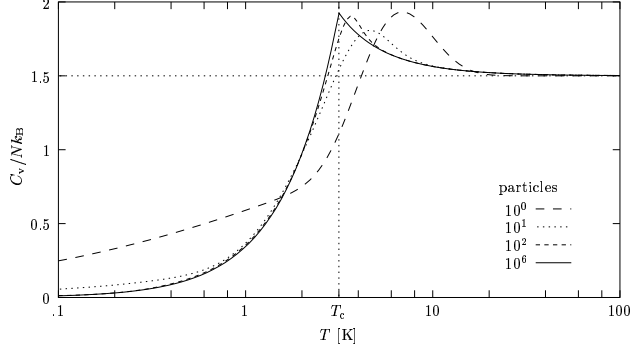
Figure 4: Heat capacity for different numbers of particles.

(19) in the form:

$$Z = \sum_{l=1}^{\infty} \frac{V^l}{l!} \sum_{w_1,\ldots,w_l=1}^{\infty} \left[\eta^{w_1+1} G_{w_1}(0)/w_1 \right] \cdots \left[\eta^{w_l+1} G_{w_l}(0)/w_l \right], \qquad (37)$$

where the summation over l arises after expanding the exponential function in a Taylor series; l physically denotes the number of loops. We omitted the constant term corresponding to $l = 0$. It is helpful to introduce the constraint $\sum_{j=1}^{l} w_j = N$ together with an extra summation over N, so that

$$\sum_{w_1,\ldots,w_l=1}^{\infty} \rightarrow \sum_{N=1}^{\infty} \sum_{\substack{w_1,\ldots,w_l=1 \\ \sum_j w_j = N}}^{\infty} \qquad (38)$$

in Eq. (37). We next group together factors with the same winding number. For given l, the right-hand side of Eq. (37) has l factors, which can be permuted in $l!$ possible ways. If a permutation contains r_j identical factors with winding number j, we have to divide by $r_j!$ because they are indistinguishable. Consequently,

$$Z = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{N=1}^{\infty} \sum_{\substack{r_1,\ldots,r_N=0 \\ \sum_j r_j = l \\ \sum_j j r_j = N}}^{\infty} \frac{l!}{\prod_j r_j!} \left[V\eta^{1+1} G_1(0)/1 \right]^{r_1} \cdots \left[V\eta^{N+1} G_N(0)/N \right]^{r_N}$$

$$= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\substack{r_1,\ldots,r_N=0 \\ \sum_j j r_j = N}}^{\infty} \frac{N!}{\prod_j j^{r_j} r_j!} \left[V\eta^{1+1} G_1(0) \right]^{r_1} \cdots \left[V\eta^{N+1} G_N(0) \right]^{r_N}, \quad (39)$$

where in the last equation we carried out the sum over l thereby losing the constraint $\sum_j r_j = l$, and inserted unity $1 = N!/N!$. The connection with Feynman's path-
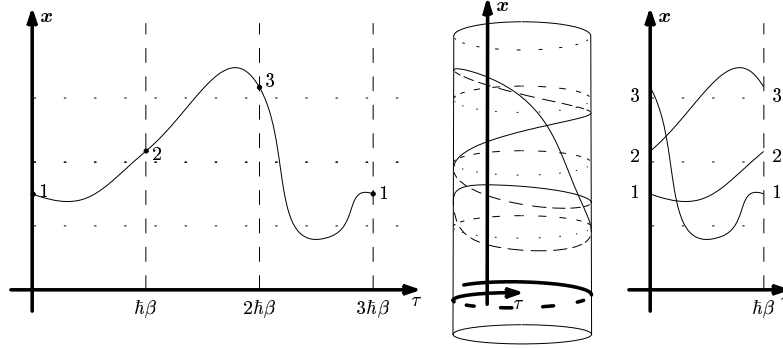
9

Figure 5: Interpretation of a loop with winding number $w = 3$ as the worldlines of three cyclicly permuted particles.

integral representation of Z is established by noting that a propagator with winding number $j > 1$ can be represented as a product of j propagators each of unit winding number:

$$V G_j(0) = \int d^3x_1 \int d^3x_2 \cdots \int d^3x_j \ G_1[(0, \mathbf{x}_1), (\hbar\beta, \mathbf{x}_2)] \tag{40}$$
$$\times G_1[(\hbar\beta, \mathbf{x}_2), (2\hbar\beta, \mathbf{x}_3)] \cdots G_1[((j-1)\hbar\beta, \mathbf{x}_j), (j\hbar\beta, \mathbf{x}_1)],$$

where we inserted $j - 1$ intermediate points $\mathbf{x}_2 \cdots \mathbf{x}_j$, and integrated over them so as to allow for all possible values of the intermediate positions. The integral $\int d^3x_1$ yields the volume factor at the left-hand side of Eq. (40). Owing to the periodicity in Euclidean time τ with period $\hbar\beta$, Eq. (40) may be equivalently written as:

$$V G_j(0) = \int d^3x_1 \int d^3x_2 \cdots \int d^3x_j \ G_1[(0, \mathbf{x}_1), (\hbar\beta, \mathbf{x}_2)] \tag{41}$$
$$\times G_1[(0, \mathbf{x}_2), (\hbar\beta, \mathbf{x}_3)] \cdots G_1[(0, \mathbf{x}_j), (\hbar\beta, \mathbf{x}_1)].$$

Physically, as was first observed by Montroll and Ward [13], a single loop with winding number j is reinterpreted as describing j particles which after a time $\hbar\beta$ are being cyclicly permuted: $1 \to 2$, $2 \to 3$, $\cdots$, $j \to 1$ (see Fig. 5).

Amusingly, this picture is reminiscent of Wheeler's attempt to explain why all electrons have the same mass and charge by supposing that their worldlines form a single huge knot [20]. When the knot is cut by a hyperplane of fixed time, many worldlines corresponding to many electrons would appear. The flaw in Wheeler's argument is—as was pointed out to him by Feynman—that to each worldline going forward in time there should be one going backward in time, corresponding to a positron. This would imply an equal number of electrons and positrons, which is not realized in nature. Our picture is not nullified by this critique because, owing to the periodicity in τ, the worldlines discussed here always move forward in time.

We recognize in Eq. (39) apart from the identity permutation $G_1(0)$, an interchange of two particles $G_2(0)$, a three-particle cyclic permutation $G_3(0)$, and so on. Since
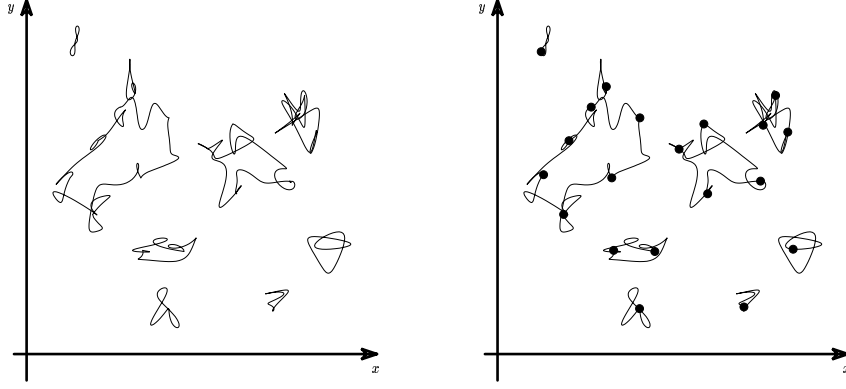
Figure 6: Connection between the string (left panel) and cycle representation (right panel) of the partition function.

$\sum_j jr_j = N$, the integer N, which was introduced in Eq. (38), denotes the total number of particles present in the cycles. Figure 6 gives a two-dimensional illustration of how the loop and cycle interpretation are connected. In the left panel, traces of eight loops in the xy-plane are depicted, each of which is parameterized by the Euclidean time τ. A closer inspection reveals that, for example, the longest loop has winding number $w = 7$. In the right panel, a filled circle is drawn whenever a time $\hbar\beta$ elapsed. They mark the starting point of a random walk executed by one particle during the time $0 \leq \tau \leq \hbar\beta$ and the end point of one executed by another particle. In the longest loop, 7 particles are cyclicly permuted in this way.

Now, any permutation of N elements can be factored into cycles of length r_j ($j = 1, 2, \cdots, N$). The number $M(r_1, r_2, \cdots, r_N)$ of such permutations is given by [8]

$$M(r_1, r_2, \cdots, r_N) = \frac{N!}{\prod_j j^{r_j} r_j!}, \tag{42}$$

which is precisely the combination appearing in (39). In this way, we arrive at the well-known path-integral representation of the partition function due to Feynman [6, 8]:

$$Z = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\sigma \in S(N)} \mathrm{sgn}^{\eta'}(\sigma) \int \mathrm{d}^3 x_1 \cdots \mathrm{d}^3 x_N G_1[(0, \mathbf{x}_{\sigma(1)}), (\hbar\beta, \mathbf{x}_{\sigma(2)})] \tag{43}$$
$$\times G_1[(0, \mathbf{x}_{\sigma(2)}), (\hbar\beta, \mathbf{x}_{\sigma(3)})] \cdots G_1[(0, \mathbf{x}_{\sigma(N)}), (\hbar\beta, \mathbf{x}_{\sigma(1)})],$$

where the sum σ is over all permutations $S(N)$ of N particles, and

$$\eta' = \frac{1}{2}(1 - \eta) = \begin{cases} 0 & \text{for bosons} \\ 1 & \text{for fermions}. \end{cases} \tag{44}$$

(A similar representation of Z, but this time not involving path integrals, was obtained earlier by Matsubara [21].) We thus have established the relation between the string

11

picture and Feynman's, where BEC is connected with the appearance of large rings involving the cyclic permutation of many particles.

In conclusion, we have described a nonrelativistic Bose gas as a path integral over spacetime loops. The loops are characterized by their string tension and the number of times they wind around the imaginary time axis. The winding number of a given loop corresponds to the number of cyclicly permuted particles in a ring in Feynman's theory. The string tension, which is determined by the chemical potential, exhibits critical behavior at T_c. Owing to the vanishing of the string tension, the loops proliferate, meaning that for an infinite system, loops with arbitrary large winding numbers appear [9]. Particles contained in long loops were shown to be condensed in the ground state.

References

[1] H. M. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).

[2] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[3] K. Huang, *Statistical Mechanics* (Wiley, New York, 1987).

[4] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).

[5] L. S. Brown. *Quantum Field Theory* (Cambridge University Press, Cambridge, 1992).

[6] R. P. Feynman, Phys. Rev. **90**, 1116 (1953); *ibid* **91**, 1291 (1953).

[7] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).

[8] R. P. Feynman, *Statistical Mechanics* (Benjamin, Reading, 1972).

[9] M. Stone, Int. J. Mod. Phys. B **4**, 1465 (1990).

[10] D. Chandler and P. G. Wolynes, J. Chem. Phys. **74**, 4078 (1981).

[11] D. M. Ceperley and E. L . Pollock, Phys. Rev. Lett. **56**, 351 (1986).

[12] D. M. Ceperley, Rev. Mod. Phys. **67**, 279 (1995).

[13] E. W. Montroll and J. C. Ward, Phys. Fluids **1**, 55 (1958).

[14] N. D. Antunes, L. M. A. Bettencourt, and M. Hindmarsh, Phys. Rev. Lett. **80**, 908 (1998); N. D. Antunes and L. M. A. Bettencourt, Phys. Rev. Lett. **81**, 3083 (1998)

[15] A. K. Nguyen and A. Sudbø, *Topological phase fluctuations, amplitude fluctuations, and criticality in extreme type-II superconductors*, preprint (1999).

[16] H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989) Vol. 1.

[17] R. J. Rivers, *Path Integrals in Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[18] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[19] E. Copeland, D. Haws, S. Holbraad, and R. Rivers, in: *The Formation and Evolution of Cosmic Strings*, edited by G. W. Gibbons, S. W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, 1990).

[20] S. S. Schweber, Rev. Mod. Phys. **58**, 449 (1986).

[21] T. Matsubara, Prog. Theor. Phys. **6**, 714 (1951).